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Matching between Piecewise Similar Curve Images

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Abstract: Matching between curve images in two dimensions is frequently performed in shape analysis. We concentrate on a specific but meaningful deformation of curve images defined by a piecewise similar relation. We present a curve matching algorithm for dealing with the deformation, together with a way of sampling points from each curve image. Our algorithm is unique in that it considers not only matching between curve images, but also sampling points. Using several experiments, we explain how to implement the algorithm for digital images of line drawings, and show that it is effective even when the number of sample points is relatively small.

Keywords: shape analysis, curve matching, line drawing recognition

1 Introduction

Matching between curve images in two dimensions is often performed in shape analysis for digital image processing, line drawing interpretation, and character handwriting recognition. A curve image is represented as a set of points. The number of points in the set can be very large, and hence, to reduce the computational cost of processing the image, a curve image is usually re-parameterized as a reduced set of sampled points [1–7]. In this case, curve matching involves finding correspondences between the sampled points of two curve images. Shape analysis relies on these correspondences.

A number of curve (or shape) matching algorithms have been proposed [1–9]. The difference between these algorithms lies in their choice of matching cost function (MCF) for matching curve images. The MCF for curve images is used to quantify dissimilarities between the images by exploiting some of their geometric attributes. Almost all the MCFs used in the curve matching algorithms are designed to be somewhat effective for certain kinds of deformations. However, which MCF to select for a particular application of curve matching remains a puzzle, since the MCFs are neither optimal nor have theoretical guarantees for all the kinds of deformations with respect to curve matching. Although the MCFs will be practically meaningful, here we are not concerned with considering certain of the kinds of deformations concurrently. In this paper, we concentrate on a specific but meaningful deformation defined by a piecewise similar relation. We present a curve matching algorithm with a novel MCF, together with a way of sampling points from each curve image. Unlike most algorithms with sample points, such as [1–6, 10], our algorithm is also unique in that it considers not only matching between curve images, but also sampling points. The algorithm has an asymptotic guarantee for finding correspondences from the sample points of a curve image to those of a piecewise similar deformation thereof. The guarantee will act as a useful guide in judging whether or not the algorithm is appropriate for an application. Using several experiments, we explain concretely how to implement the algorithm for digital images, and show that it is effective even when the number of sample points is relatively small.

The organization of this paper is as follows. We introduce the piecewise similar relation, and formulate curve matching in Section 2. We describe our curve matching algorithm in Section 3. The experimental results of the algorithm are shown in Section 4. We conclude with a summary in Section 5.

2 Preliminaries

Let $\mathbb{Z}$ be the integers and $\mathbb{R}$ the real numbers. The nonnegative and positive elements in $\mathbb{Z}$ are denoted by $\mathbb{Z}_0^+$ and $\mathbb{Z}^+$, respectively. For any $i, j \in \mathbb{Z}$, $\mathbb{Z}_i^j$ denotes the integers from i to j. The nonnegative and positive elements in $\mathbb{R}$ are denoted by $\mathbb{R}_0^+$ and $\mathbb{R}^+$, respectively. $\sharp\,(\cdot)$ represents the number of elements in a finite set, and $\|\cdot\|$ denotes the

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norm of a vector in Euclidean space.

2.1 Piecewise Similar Relation

We define a curve, curve segment, and piecewise regular curve to introduce a similarity of curve images.

Definition 1 (Curve)**.** *Let $I = [a, b] \subset \mathbb{R}$ be a closed interval, where $a < b$. A plane curve is a continuous map $C_I : I \to \mathbb{R}^2$, with*

$$C_I(t) \triangleq (x_I(t), y_I(t)). \tag{1}$$

When a time-parameter $t \in I$ increases from a to b, we obtain the directed trajectory of $C_I(t)$,

$$C_I(I) \triangleq \{ C_I(t) \mid t \in I \} , \tag{2}$$

where the ordering of points in the curve image $C_I(I)$ preserves that of t in I, that is, for all t, $t' \in I$ where $t < t'$, $C_I(t)$ precedes $C_I(t')$ in $C_I(I)$. The curve image which is an ordered set of points with respect to t is simply called an image. A plane curve C_I that

1. *is twice differentiable on $(a, b) \subset I$, and*
2. *satisfies $dC_I(t)/dt \neq 0$ for all $t \in (a, b)$,*

is said to be regular, and its image is called a regular image.

Definition 2 (Curve Segment)**.** *For any interval $[a, b] \subseteq I$, the segment of curve C_I with respect to $[a, b]$ is described as a continuous map $C_I \,|[a, b] : [a, b] \to \mathbb{R}^2$. The image of a segment is also called an image.*

Definition 3 (Piecewise Regular Curve)**.** *Let C_I be a curve for any $I = [a, b]$. If there exists a partition of I,*

$$a = k_0 < k_1 < \cdots < k_{N-1} < k_N = b, \tag{3}$$

such that

1. *N is a finite integer, and*
2. *segment $C_I \,|[k_i, k_{i+1}]$ is regular for all $i \in \mathbb{Z}_0^{N-1}$,*

then C_I is called a piecewise regular curve and $C_I(I)$ is called a piecewise regular image.

The total length of a piecewise regular image is calculated as the sum of all the segment image lengths.

Definition 4 (Image Set)**.** *The set of piecewise regular images with positive length is denoted as $\mathfrak{S}$.*

When an image is uniformly magnified or reduced, the resulting image is similar to the original image in the following sense.

Definition 5 (Similarity)**.** *Let $C_I(I)$ and $C'_J(J)$ be any images in $\mathfrak{S}$. If there exist a map $\zeta : C_I(I) \to C'_J(J)$ and a constant $\lambda \in \mathbb{R}^+$ such that, for all c_1, $c_2 \in C_I(I)$,*

$$\| \zeta(c_1) - \zeta(c_2) \| = \lambda \| c_1 - c_2 \| , \tag{4}$$

then $C_I(I)$ and $C'_J(J)$ are similar images and we write $C_I(I) \sim C'_J(J)$.

Similarity plays an important role in human recognition of images, because similar images appear to have the same shape, even though they may differ in scale. For example, a small image of the letter "S" and a large image thereof are recognized as the same letter. Analogous to similarity is piecewise similarity according to Definition 6. It plays the same role as similarity in human recognition.

Definition 6 (Piecewise Similarity)**.** *Let $C_I(I)$ and $C'_J(J)$ be images in $\mathfrak{S}$, where $I = [a, b]$ and $J = [a', b']$. If there exist partitions of I and J,*

$$a = k_0 < k_1 < \cdots < k_{N-1} < k_N = b, \tag{5}$$

$$a' = l_0 < l_1 < \cdots < l_{N-1} < l_N = b', \tag{6}$$

such that

1. *N is a finite integer,*
2. *for all $i \in \mathbb{Z}_0^{N-1}$, $C_I \,|[k_i, k_{i+1}] \,([k_i, k_{i+1}])$ and $C'_J \,|[l_i, l_{i+1}] \,([l_i, l_{i+1}])$ are regular images in $\mathfrak{S}$, and*
3. *for all $i \in \mathbb{Z}_0^{N-1}$,*

$$C_I \,|[k_i, k_{i+1}] \,([k_i, k_{i+1}]) \sim C'_J \,|[l_i, l_{i+1}] \,([l_i, l_{i+1}]) , \tag{7}$$

then $C_I(I)$ and $C'_J(J)$ are piecewise similar and we write $C_I(I) \overset{\mathrm{P}}{\sim} C'_J(J)$. The points $C_I(k_0), \ldots, C_I(k_N)$ are called segment endpoints on $C_I(I)$.

Example 1 (Piecewise Similarity)**.** *On the left in Fig. 1 is an image of the letter "S". In the center of the figure, the original image has been deformed by uniformly making the upper part of the letter smaller, while on the right, the image has been further deformed by uniformly making the lower part larger. Accordingly, these images are piecewise similar to each other and can be recognized by humans as representing the same letter "S".*

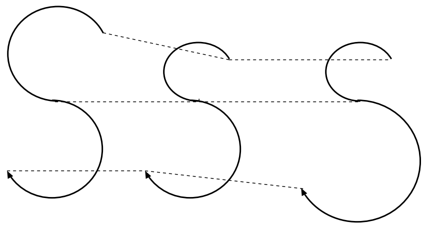


Fig. 1: Piecewise similar deformation of images.

Most of the raw data available from a database of shapes, line drawings, and characters are not drawn to scale. If some of the raw data in the same classes have a similar relation, it is relatively easy to make them the same size by preprocessing and then to find the correspondences between them. However, we have rarely seen such data in practice. Most of the data have a piecewise similar relation, since they appear to represent the same shape. In general, it is difficult to find correspondences between them. Accordingly, in this paper, we concentrate on the piecewise similar deformation of images.

2.2 Curve Matching

A curve image is often re-parameterized as a reduced set of sampled points. This is described with Definitions 7 and 8.

Definition 7 (Sample Points)**.** *For any interval $I = [a, b]$ and any $N \in \mathbb{Z}^+$, let*

$$\gamma_N(I) \triangleq \{ \{ t_0, t_1, \ldots, t_{N-1}, t_N \} \in I^{N+1} \mid a = t_0 < t_1 < \cdots < t_{N-1} < t_N = b \}. \quad (8)$$

For any sequence $T_N = \{ t_0, \ldots, t_N \} \in \gamma_N(I)$,

$$C_I(T_N) \triangleq \{ C_I(t_i) \in C_I(I) \mid i \in \mathbb{Z}_0^N \}, \quad (9)$$

are called the sample points of $C_I(I)$. The ordering of the sample points in $C_I(T_N)$ preserves the ordering of $t_i \in I$.

Definition 8 (Re-parameterization)**.** *We define*

$$\Gamma_N(C_I(I)) \triangleq \left\{ C_I(T_N) \in C_I(I)^{N+1} \mid T_N \in \gamma_N(I) \right\}. \quad (10)$$

For any sequence $T_N = \{ t_0, \ldots, t_N \} \in \gamma_N(I)$, the sample points on the image are simply denoted as

$$P_N \triangleq C_I(T_N). \quad (11)$$

For all $i \in \mathbb{Z}_0^N$, the i-th element of P_N is denoted by

$$p_i \triangleq C_I(t_i), \quad (12)$$

and the components of the i-th element are expressed as

$$(x_i, y_i) \triangleq p_i. \quad (13)$$

For all $i \in \mathbb{Z}_0^{N-1}$, the finite difference at p_i is defined as

$$\Delta p_i = (\Delta x_i, \Delta y_i), \quad (14)$$
$$\triangleq (x_{i+1} - x_i, y_{i+1} - y_i). \quad (15)$$

For all $i \in \mathbb{Z}_0^{N-2}$, the second-order finite difference at p_i is expressed as

$$\Delta^2 p_i = \left(\Delta^2 x_i, \Delta^2 y_i \right), \quad (16)$$
$$\triangleq (\Delta x_{i+1} - \Delta x_i, \Delta y_{i+1} - \Delta y_i). \quad (17)$$

For all $i \in \mathbb{Z}_0^{N-1}$, the unit tangent and unit normal vectors at p_i are defined as

$$e_{P_N}^{(1)}(p_i) \triangleq \left(\frac{\Delta x_i}{\| \Delta p_i \|}, \frac{\Delta y_i}{\| \Delta p_i \|} \right), \quad (18)$$
$$e_{P_N}^{(2)}(p_i) \triangleq \left(-\frac{\Delta y_i}{\| \Delta p_i \|}, \frac{\Delta x_i}{\| \Delta p_i \|} \right), \quad (19)$$

respectively. For all $i \in \mathbb{Z}_0^{N-2}$, the curvature at p_i is defined as

$$\kappa_{P_N}(p_i) \triangleq \frac{\Delta x_i \Delta^2 y_i - \Delta^2 x_i \Delta y_i}{\| \Delta p_i \|^3}. \quad (20)$$

For simplicity, an image $C_I(I)$ is denoted by $\mathcal{C}$. Also, we sometimes describe the unit vectors using angles.

Definition 9 (Angle)**.** *Let $\mathcal{C}$ be an image in $\mathfrak{S}$. For any $P_N \in \Gamma_N(\mathcal{C})$, we define θ_{P_N} such that for all $i \in \mathbb{Z}_0^{N-1}$, the unit tangent and unit normal vectors at $p_i \in P_N$ are*

$$e_{P_N}^{(1)}(p_i) = (\cos\theta_{P_N}(p_i), \sin\theta_{P_N}(p_i)), \quad (21)$$
$$e_{P_N}^{(2)}(p_i) = (-\sin\theta_{P_N}(p_i), \cos\theta_{P_N}(p_i)). \quad (22)$$

For all $i \in \mathbb{Z}_0^{N-1}$, the finite difference at p_i is defined as

$$\Delta\theta_{P_N}(p_i) \triangleq \theta_{P_N}(p_{i+1}) - \theta_{P_N}(p_i). \quad (23)$$

Now, curve matching is formulated using sample points.

Definition 10 (Curve Matching)**.** *Let $\mathcal{C}$ and $\mathcal{C}'$ be images in $\mathfrak{S}$. We say that $\mathcal{C}$ matches $\mathcal{C}'$ with $P_N \in \Gamma_N(\mathcal{C})$ and $Q_M \in \Gamma_M(\mathcal{C}')$, respectively, if there is a correspondence from each element in P_N to an element in Q_M. A correspondence is represented by a many-to-one map $f : \mathbb{Z}_0^N \to \mathbb{Z}_0^M$ that satisfies the following two conditions:*

1. *$f(0) = 0$ and $f(N) = M$, and*

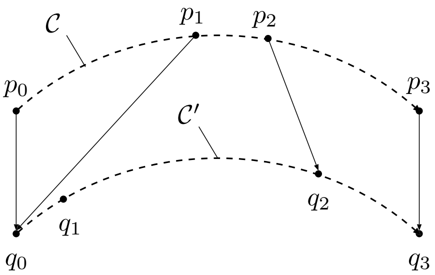


Fig. 2: Matching map f.

2. $f(i) \leq f(i+1)$ *for all* $i \in \mathbb{Z}_0^{N-1}$,

where $f(i) = j$ *denotes the correspondence from* $p_i \in P_N$ *to* $q_j \in Q_M$. *This is called a matching map.*

Example 2 (Matching Map)**.** *Fig. 2 illustrates a matching map. Let* $\mathcal{C}$ *and* $\mathcal{C}'$ *denote the upper and lower curve images in Fig. 2, respectively. The points on the curve images represent the sample points on the images. The arrows depict correspondences from the sample points on* $\mathcal{C}$ *to those on* $\mathcal{C}'$*, which are expressed as* $f(0) = 0$, $f(1) = 0$, $f(2) = 2$, *and* $f(3) = 3$.

3 Curve Matching Algorithm

We start with a definition for equipartition sample points.

Definition 11 (Equipartition Sample Points)**.** *Let* $\mathcal{C}$ *be an image in* $\mathfrak{S}$. *Let* p_i *denote the* i*-th element of* $P_N \in \Gamma_N(\mathcal{C})$. *If for all* $i \in \mathbb{Z}_0^{N-1}$, *the finite difference at* p_i *satisfies*

$$\| \Delta p_i \| = r_N > 0, \tag{24}$$

then P_N *is referred to as the equipartition sample points on* $\mathcal{C}$. *For any* $N \in \mathbb{Z}^+$, *the set of such sample points on* $\mathcal{C}$ *is simply denoted as*

$$\Gamma_N^*(\mathcal{C}) \triangleq \left\{ P_N \in \mathcal{C}^{N+1} \;\middle|\; \| \Delta p_i \| = r_N, i \in \mathbb{Z}_0^{N-1} \right\}. \tag{25}$$

Note that r_N depends only on N, and not on i. The following curvature-based measure plays an important role in quantifying the difference between images in terms of piecewise similarity.

Definition 12 (Curvature-based Measure)**.** *Let* $\mathcal{C}$ *be an image in* $\mathfrak{S}$. *Let* $\underline{\mathcal{C}} \subseteq \mathcal{C}$ *denote a part of the image in* $\mathfrak{S}$. *For any* $P_N \in \Gamma_N(\mathcal{C})$, *the measure* $\alpha_{P_N} : \mathfrak{S} \to \mathbb{R}$ *is defined as*

$$\alpha_{P_N}(\underline{\mathcal{C}}) \triangleq \sum_{p_i \in \underline{\mathcal{C}} \cap P_N,\, i \in \mathbb{Z}_0^{N-2}} \kappa_{P_N}(p_i), \tag{26}$$

where κ_{P_N} *is the curvature defined in* (20).

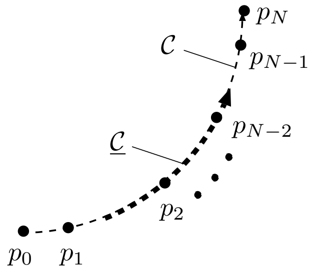


Fig. 3: Part of an image and sample points.

Example 3 (Curvature-based Measure)**.** *Fig. 3 depicts a part* $\underline{\mathcal{C}}$ *of an image* $\mathcal{C}$ *and sample points* P_N *according to Definition 12. In this case, because*

$$\underline{\mathcal{C}} \cap P_N = \{ p_2, \ldots, p_{N-2} \}, \tag{27}$$

the measure $\alpha_{P_N}(\underline{\mathcal{C}})$ *is calculated as*

$$\alpha_{P_N}(\underline{\mathcal{C}}) = \kappa_{P_N}(p_2) + \cdots + \kappa_{P_N}(p_{N-2}). \tag{28}$$

We introduce a convenient notation to indicate a part of an image together with its sample points.

Definition 13 (Image with Sample Points)**.** *Let* $\mathcal{C}$ *be an image in* $\mathfrak{S}$. *For any* p_i *and* $p_{i'}$ *in* $P_N \in \Gamma_N(\mathcal{C})$, $\mathcal{C}\,|[p_i, p_{i'}]$ *denotes a part of* $\mathcal{C}$ *such that*

1. *it exists in* $\mathfrak{S}$,

2. *it contains all sample points between* p_i *and* $p_{i'}$, *but does not include the other elements of* P_N.

We should note that $\mathcal{C}\,|[p_i, p_i]$ contains a single sample point p_i, but its length is positive because it is in $\mathfrak{S}$, and that for all images with the same sample points, the curvature-based measure gives the same value.

Example 4 (Image with Sample Points)**.** *The part* $\underline{\mathcal{C}}$ *of* $\mathcal{C}$ *in Fig. 3 can be expressed as* $\mathcal{C}\,|[p_2, p_{N-2}]$.

Definition 14 (Dissimilarity Measure)**.** *Let* $\mathcal{C}$ *and* $\mathcal{C}'$ *be images in* $\mathfrak{S}$. *Let* $\underline{\mathcal{C}} \subseteq \mathcal{C}$ *and* $\underline{\mathcal{C}}' \subseteq \mathcal{C}'$ *denote their parts in* $\mathfrak{S}$. *For any* $P_N \in \Gamma_N(\mathcal{C})$ *and* $Q_M \in \Gamma_M(\mathcal{C}')$, *their dissimilarity measure* $\mu_{P_N,Q_M} : \mathfrak{S} \times \mathfrak{S} \to \mathbb{R}_0^+$ *is defined as*

$$\mu_{P_N,Q_M}\left(\underline{\mathcal{C}}, \underline{\mathcal{C}}'\right) \triangleq \left| \alpha_{P_N}(\underline{\mathcal{C}}) - \alpha_{Q_M}\left(\underline{\mathcal{C}}'\right) \right|. \tag{29}$$

It is simple to compute the dissimilarity measure, since it requires only the curvatures. The dissimilarity measure is invariant for all translations, reflections, and rotations, since the curvatures in α_{P_N} and α_{Q_M} are invariant for these. Using the dissimilarity measure, we describe an MCF that computes the cost of obtaining correspondences from the sample points of one image to those of another.

Definition 15 (Matching Cost Function)**.** *Let $\mathcal{C}$ and $\mathcal{C}'$ be images in $\mathfrak{S}$. For any N, $M \in \mathbb{Z}^+$, let $P_N \in \Gamma_N(\mathcal{C})$ and $Q_M \in \Gamma_M(\mathcal{C}')$ denote the respective sample points. For any matching map $f : \mathbb{Z}_0^N \to \mathbb{Z}_0^M$, let*

$$\underline{\mathcal{I}}_f \triangleq \bigcup_{i=0}^{N} \left\{ \min_{i' \in \mathcal{I}_f(i)} i' \right\}, \quad \overline{\mathcal{I}}_f \triangleq \bigcup_{i=0}^{N} \left\{ \max_{i' \in \mathcal{I}_f(i)} i' \right\}, \tag{30}$$

where for all $i \in \mathbb{Z}_0^N$,

$$\mathcal{I}_f(i) \triangleq \left\{ i' \in \mathbb{Z}_0^N \mid f(i) = f(i') \right\}. \tag{31}$$

Since $\sharp\left(\underline{\mathcal{I}}_f\right) = \sharp\left(\overline{\mathcal{I}}_f\right)$ holds, let $L = \sharp\left(\underline{\mathcal{I}}_f\right) = \sharp\left(\overline{\mathcal{I}}_f\right)$. For all $n \in \mathbb{Z}_0^{L-1}$, let $\underline{i}_n$ and $\overline{i}_n$ be the $(n+1)$-th smallest elements in $\underline{\mathcal{I}}_f$ and $\overline{\mathcal{I}}_f$, respectively. Given a matching map $f : \mathbb{Z}_0^N \to \mathbb{Z}_0^M$, the matching cost function (MCF) for $\mathcal{C}$ and $\mathcal{C}'$ under P_N and Q_M is described as

$$d_{P_N,Q_M}\left(\mathcal{C},\mathcal{C}' \mid f\right) \triangleq \sum_{n=0}^{L-1} \mu_{P_N,Q_M}\left(\mathcal{C}\left|\left[p_{\underline{i}_n}, p_{\overline{i}_n}\right]\right., \mathcal{C}'\left|\left[q_{j_{n-1}+1}, q_{j_n}\right]\right.\right), \tag{32}$$

where index j_n is defined as

$$j_n \triangleq \begin{cases} -1, & \text{if } n = -1, \\ f\left(\overline{i}_n\right), & \text{otherwise}. \end{cases} \tag{33}$$

The best matching maps are described as

$$f^* \triangleq \operatorname*{argmin}_f d_{P_N,Q_M}\left(\mathcal{C},\mathcal{C}' \mid f\right). \tag{34}$$

Example 5 (Matching Cost Function)**.** *Consider the matching map f given in Example 2. According to* (31)*, we have $\mathcal{I}_f(0) = \{0,1\}$, $\mathcal{I}_f(1) = \{0,1\}$, $\mathcal{I}_f(2) = \{2\}$ and $\mathcal{I}_f(3) = \{3\}$. Hence, $\underline{\mathcal{I}}_f = \{0,2,3\}$ and $\overline{\mathcal{I}}_f = \{1,2,3\}$. The matching cost for the matching map f is written as*

$$\begin{aligned} d_{P_3,Q_3}\left(\mathcal{C},\mathcal{C}' \mid f\right) &= \mu_{P_3,Q_3}\left(\mathcal{C}\left|[p_0,p_1]\right., \mathcal{C}'\left|[q_0,q_0]\right.\right) \\ &+ \mu_{P_3,Q_3}\left(\mathcal{C}\left|[p_2,p_2]\right., \mathcal{C}'\left|[q_1,q_2]\right.\right) \\ &+ \mu_{P_3,Q_3}\left(\mathcal{C}\left|[p_3,p_3]\right., \mathcal{C}'\left|[q_3,q_3]\right.\right). \end{aligned} \tag{35}$$

We now describe our algorithm incorporating the MCF.

Algorithm 1 (Curve Matching)**.** *Perform the steps given below.*

1. *Extract sample points P_N and Q_M from images $\mathcal{C}$ and $\mathcal{C}'$, respectively, such that constraints 1a, 1b, and 1c given below hold:*
   (a) *$P_N = \{p_0, \ldots, p_N\} \in \Gamma_N^*(\mathcal{C})$,*
   (b) *$Q_M = \{q_0, \ldots, q_M\} \in \Gamma_M^*(\mathcal{C}')$, and*
   (c) *for any $\epsilon \in \mathbb{R}^+$, there exist $N_0 \in \mathbb{Z}^+$ and $M_0 \in \mathbb{Z}^+$ such that for all $i \in \mathbb{Z}_0^{N-1}$, all $j \in \mathbb{Z}_0^{M-1}$, all $N \geq N_0$ and all $M \geq M_0$,*
   $$\left| \frac{1}{\|\Delta p_i\|} - \frac{1}{\|\Delta q_j\|} \right| < \epsilon. \tag{36}$$
2. *Using P_N and Q_M obtained in the previous step, find the best matching maps f^* that give the minimum cost, using a search algorithm.*
3. *Express correspondences from P_N to Q_M, according to f^*.*

Although the MCF is simple, it is sufficient for our algorithm to find correspondences between piecewise similar curve images as shown in Theorem 1 and Corollary 1.

Theorem 1. *Let $\mathcal{C}$ and $\mathcal{C}'$ be images in $\mathfrak{S}$. Let $\underline{\mathcal{C}} \subseteq \mathcal{C}$ and $\underline{\mathcal{C}}' \subseteq \mathcal{C}'$ denote the respective parts which are regular images in $\mathfrak{S}$. If*

1. *$\underline{\mathcal{C}} \sim \underline{\mathcal{C}}'$,*
2. *sample points P_N and Q_M are extracted from $\mathcal{C}$ and $\mathcal{C}'$, respectively, such that they satisfy constraints 1a, 1b, and 1c of the algorithm, and*
3. *N and M go to infinity such that for all $i \in \mathbb{Z}_0^{N-1}$,*
   $$\lim_{N,\, M \to \infty} \left| \frac{\nu_{P_N}\left(\underline{\mathcal{C}}\right) - \nu_{Q_M}\left(\underline{\mathcal{C}}'\right)}{\|\Delta p_i\|} \right| = 0, \tag{37}$$
   where p_i denotes the i-th point of P_N, and $\nu_{P_N} : \mathfrak{S} \to \mathbb{R}$ is defined by
   $$\nu_{P_N}\left(\underline{\mathcal{C}}\right) \triangleq \sum_{p_i \in \underline{\mathcal{C}} \cap P_N,\, i \in \mathbb{Z}_0^{N-2}} \Delta\theta_{P_N}(p_i), \tag{38}$$

then

$$\lim_{N,\, M \to \infty} \mu_{P_N,Q_M}\left(\underline{\mathcal{C}}, \underline{\mathcal{C}}'\right) = 0. \tag{39}$$

The proof sketch is given in [11]. This theorem states that there is an asymptotic guarantee for coping with partially similar deformations of images under the constraints, because if two image parts are similar, then their dissimilarity measure tends asymptotically to zero. The dissimilarity measure confirms whether or not images can be similar by verifying the equation in (39).

From Theorem 1, we readily obtain an asymptotic guarantee of the algorithm in Corollary 1. Interestingly, the algorithm finds the matching maps that give the minimum cost without knowing the segment endpoints or the scale of piecewise similar images in advance.

Corollary 1. *Let $\mathcal{C}$ and $\mathcal{C}'$ be any images in $\mathfrak{S}$. If $\mathcal{C} \overset{\mathrm{P}}{\sim} \mathcal{C}'$, then the algorithm finds a matching map for which the matching cost from $\mathcal{C}$ to $\mathcal{C}'$ tends to zero as $N \to \infty$ and $M \to \infty$.*

Proof. From Theorem 1, because $\mathcal{C} \overset{\mathrm{P}}{\sim} \mathcal{C}'$, there exists a matching map $f : \mathbb{Z}_0^N \to \mathbb{Z}_0^M$ for which the matching cost described in (32) tends to zero as $N \to \infty$ and $M \to \infty$. Hence, the minimum cost given by f^* tends to zero as $N \to \infty$ and $M \to \infty$. □

Recall that $\overset{\mathrm{P}}{\sim}$ denotes the piecewise similar relation (see Definition 6). Proposition 1 implies that we can simplify constraint 1c of the algorithm when images are digitized.

Proposition 1. *Let $\mathcal{C}$ and $\mathcal{C}'$ be images in $\mathfrak{S}$. Let p_i and q_j be the i-th and j-th sample points of $P_N \in \Gamma_N(\mathcal{C})$ and $Q_M \in \Gamma_M(\mathcal{C}')$, respectively. If for all $i \in \mathbb{Z}_0^{N-1}$ and all $j \in \mathbb{Z}_0^{M-1}$,*

1. *$\| \Delta p_i \| \geq 1$ and $\| \Delta q_j \| \geq 1$, and*
2. *for a given $\epsilon \in \mathbb{R}^+$,*

$$| \| \Delta p_i \| - \| \Delta q_j \| | < \epsilon, \tag{40}$$

then (36) *holds.*

The proof is routine. Note that digital images embedded in the pixel points of $\left(\mathbb{Z}_0^+\right)^2$ always satisfy the first condition of Proposition 1 if the same pixel point is not sampled more than once. In this case, Proposition 1 indicates that minimizing $| \| \Delta p_i \| - \| \Delta q_j \| |$ is sufficient to minimize $|1/ \| \Delta p_i \| - 1/ \| \Delta q_j \| |$. For the same reason, we do not need to take care the third condition (37) of Theorem 1 in implementation when images are digitized.

4 Experiments

In this section, we show experimental results of the algorithm to explain concretely how the algorithm is implemented for digital images. This is because digital images are embedded in the pixel points of $\left(\mathbb{Z}_0^+\right)^2$, but not $\mathbb{R}^2$ [12].

Line Drawing Images We have implemented the algorithm for digital images of line drawings, examples of which are shown in Figs. 4 and 5. The images in the figures were drawn by hand with a pen on a touch panel[1]. Hence, they are affected by hand oscillation and are a little distorted. Each example consists of three images of the same class. The center and right images in each example have been drawn so as to be piecewise similar to the left image. In Fig. 4, the center image has been deformed by uniformly reducing the upper part of the left image, while the right image has been further deformed by uniformly magnifying the lower part. The example shown in Fig. 5 is much more complicated in shape. In Fig. 5, the center image has been deformed by uniformly reducing the middle part of the left image, while the right image has been deformed by uniformly magnifying the starting spiral part of the left image.

The left image in each example is called the query image. The center and right images, which are deformations of the query image, are called database images. For each of the examples, we use our algorithm to obtain correspondences from the sample points of the query image to those of a database image.

In this section, let $\mathcal{C}$ be the query image and $\mathcal{C}'$ a database image. These digital images are expressed as

$$\mathcal{C} = \left\{ c_0, \ldots, c_{\overline{N}-1} \right\}, \quad \mathcal{C}' = \left\{ c'_0, \ldots, c'_{\overline{M}-1} \right\}, \tag{41}$$

where c_n and c'_m denote the n-th and m-th elements of $\mathcal{C}$ and $\mathcal{C}'$ in the pixel points, respectively, and $\overline{N}$ and $\overline{M}$ denote the number of elements in $\mathcal{C}$ and $\mathcal{C}'$, respectively. For all $n \in \mathbb{Z}_0^{\overline{N}-1}$, the length of a subset of a digital image $\mathcal{C}$ is given by

$$\sigma_n(\mathcal{C}) = \sum_{n'=0}^{n-1} \| c_{n'} - c_{n'+1} \|, \tag{42}$$

where $\sigma_0(\mathcal{C}) = 0$.

Implementation of Step 1 According to Proposition 1, we replace (36) with (40) in implementing step 1 of the algorithm. This results in the following procedure. For any $N \leq \overline{N} - 1$, when segmenting a query image $\mathcal{C}$ with equipartition sample points P_N on $\mathcal{C}$, the i-th equipartition sample point p_i of P_N is the n_i-th point c_{n_i} of $\mathcal{C}$ such that for all $i \in \mathbb{Z}_0^N$,

$$n_i = \underset{n \in \mathbb{Z}_0^{\overline{N}-1}}{\operatorname{argmin}} \left| \frac{\sigma_n(\mathcal{C})}{\sigma_{\overline{N}-1}(\mathcal{C})} - \frac{i}{N} \right|. \tag{43}$$

Thus, we extract $N + 1$ equipartition sample points from $\mathcal{C}$. In this case, because of constraint 1c rewritten as (40), the number of equipartition sample points Q_M on the other image $\mathcal{C}'$ is meant to be

$$M = \underset{z \in \mathbb{Z}^+}{\operatorname{argmin}} \left| \frac{z}{N} - \frac{\sigma_{\overline{M}-1}(\mathcal{C}')}{\sigma_{\overline{N}-1}(\mathcal{C})} \right|. \tag{44}$$

[1] A drawing software is available at http://www.prl.info.hiroshima-cu.ac.jp/~kiwata/panel/.

Then, Q_M is obtained from $\mathcal{C}'$ in the same way. Thus, we obtain P_N and Q_M which approximately satisfy the constraints in step 1 of the algorithm.

Implementation of Step 2 In step 2 of the algorithm, there appear to be some matching maps with the same cost, because the curvatures at p_{N-1} and p_N are ignored in (26). This would ordinarily be a problem for relatively small values of N. However, to avoid such a problem, we use instead

$$\alpha_{P_N}(\underline{\mathcal{C}}) \triangleq \sum_{p_i \in \underline{\mathcal{C}} \cap P_N,\, i \in \mathbb{Z}_0^N} \kappa_{P_N}(p_i), \tag{45}$$

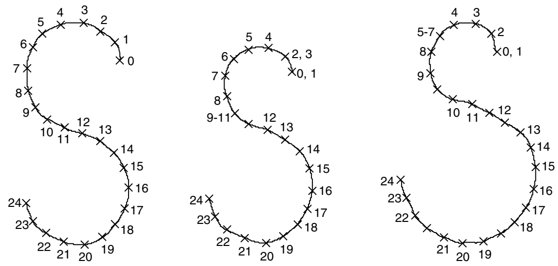
in computing the curvature-based measure in (32). Here the curvatures at p_{N-1} and p_N are computed additionally using the pseudo finite differences,

$$\Delta p_N = \frac{1}{3}\left(2\Delta p_{N-1} + \Delta p_{N-2}\right), \tag{46}$$

$$\Delta p_{N+1} = \frac{1}{5}\left(4\Delta p_{N-1} + \Delta p_{N-2}\right). \tag{47}$$

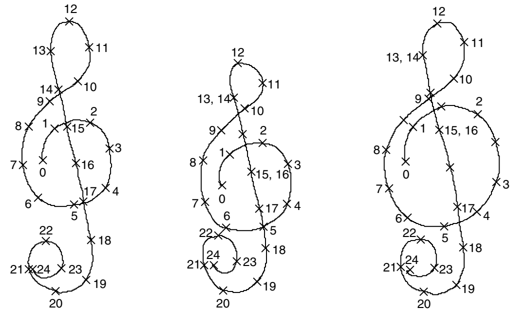
The second-order finite difference in (17) can be defined additionally for $i = N-1, N$ using these finite differences.

Results We set $N = 24$ in all the examples. This means that there are 25 equipartition sample points on the query image in each example. Recall that the number of equipartition sample points on each database image is determined according to (44) when N is given. The resulting correspondences obtained by the algorithm are shown in Figs. 4 and 5. In cases where there were several best matchings providing the minimum cost, we have shown only one of these. In the figures, an x on the image represents an equipartition sample point. In each example, the sample points on the query image are labeled with successive numbers from 0 to 24. The numbering of sample points on the database images indicates correspondences from sample points with the same numbers on the query image. For example, the sample point labeled 0 in Fig. 4(a) corresponds to the sample points labeled 0 in Figs. 4(b) and 4(c). Unnumbered sample points on a database image have no correspondence from sample points on the query image. The figures confirm that the algorithm consistently provides correct correspondences from the sample points on a query image to those on an almost piecewise similar database image. It is somewhat surprising that correct correspondences are given even for such complicated images as in Fig. 5. The results also suggest that the algorithm performs well even with a relatively small number of sample points.



(a) S 1. (b) S 2. (c) S 3.

Fig. 4: Best matching map from the query image "S 1" to the database images "S 2" and "S 3".



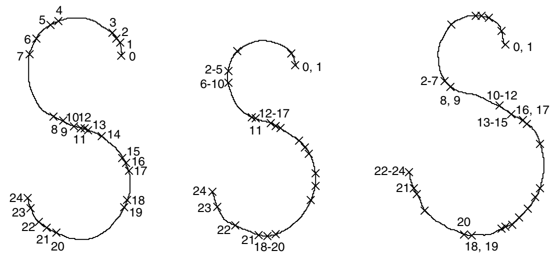
(a) G-clef 1. (b) G-clef 2. (c) G-clef 3.

Fig. 5: Best matching map from the query image "G-clef 1" to the database images "G-clef 2" and "G-clef 3".

Next, we examine the effect of the constraints of the algorithm. Instead of using equipartition sample points, we employed sample points randomly extracted from the respective images in executing the algorithm. Clearly, such samples points do not adhere to the constraints. The results of the correspondences on the same example of images as Fig. 4 are shown in Fig. 6. In the figure, an x on an image denotes a randomly extracted sample point. Comparing Figs. 4 and 6, we confirm that the algorithm failed to find the correct correspondences. It follows that the constraints provide an outstanding method for using sampling points in the algorithm. Thus, it is effective to consider both sampling points and matching.

5 Conclusion

We explained that our algorithm gives the best matchings between piecewise similar images without knowing the segment endpoints or the scale of the images in advance. The most important use for the best matchings is as a foundation for shape analysis. It may be necessary to select a few



(a) S 1. (b) S 2. (c) S 3.

Fig. 6: Best matching map from the query image "S 1" to the database images "S 2" and "S 3" using randomly extracted sample points.

of the best matchings according to application dependent properties. For example, in character handwriting recognition, we sometimes need to select matchings by examining the difference between left and right derivatives at each segment endpoint, because not all piecewise similar images represent the same character. However, even in such a case, the algorithm is still effective in retrieving a possible small set of correspondences before embarking on more accurate matching.

In this paper, we discussed a piecewise deformation given by a similarity relation. We presented a curve matching algorithm for coping with the deformation of images, together with a way of distributing sample points on the respective images. We confirmed through several experimental results that the algorithm is effective even with a relatively small number of sample points.

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