Wasserstein幾何と ϕ-正規分布族

metric geom. on {prob. meas.}

ϕ : fct

$(\phi(s) = s$: Gauss$)$

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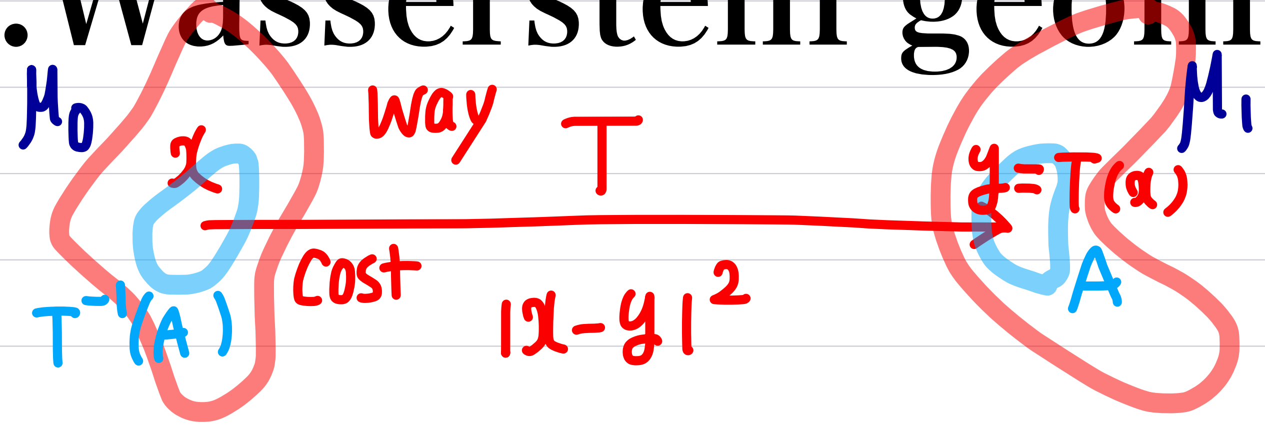
1.Wasserstein geometry



Back ground : Optimal Transport.

$$\left\{ \begin{array}{l} \mu_i : \text{prob. meas. on } \mathbb{R}^n \\ \qquad \text{finite } 2^{nd} \text{ moment, abs. conti. w.r.t. } \mathcal{L}^n \end{array} \right\} =: \mathcal{P}_2^{ac}$$

$$\{T:\mathbb{R}^n\to\mathbb{R}^n \mid \underline{\mu_1(A)=\mu_0(T^{-1}(A))\ \forall A\subseteq\mathbb{R}^n}\} =: \mathcal{T}(\mu_0,\mu_1)$$

$T_\# \mu_0 = \mu_1$ (push-forward)

$$W_2(\mu_0,\mu_1) := \underline{\min}_{T} \left(\int_{\mathbb{R}^n} |x-T(x)|^2 \, d\mu_0(x)\right)^{1/2}$$

$(\mathcal{P}_2, W_2)$: met. sp.

Thm. (Brenier 91)

$\forall \mu_0, \mu_1 \in \mathcal{P}_2^{ac}$

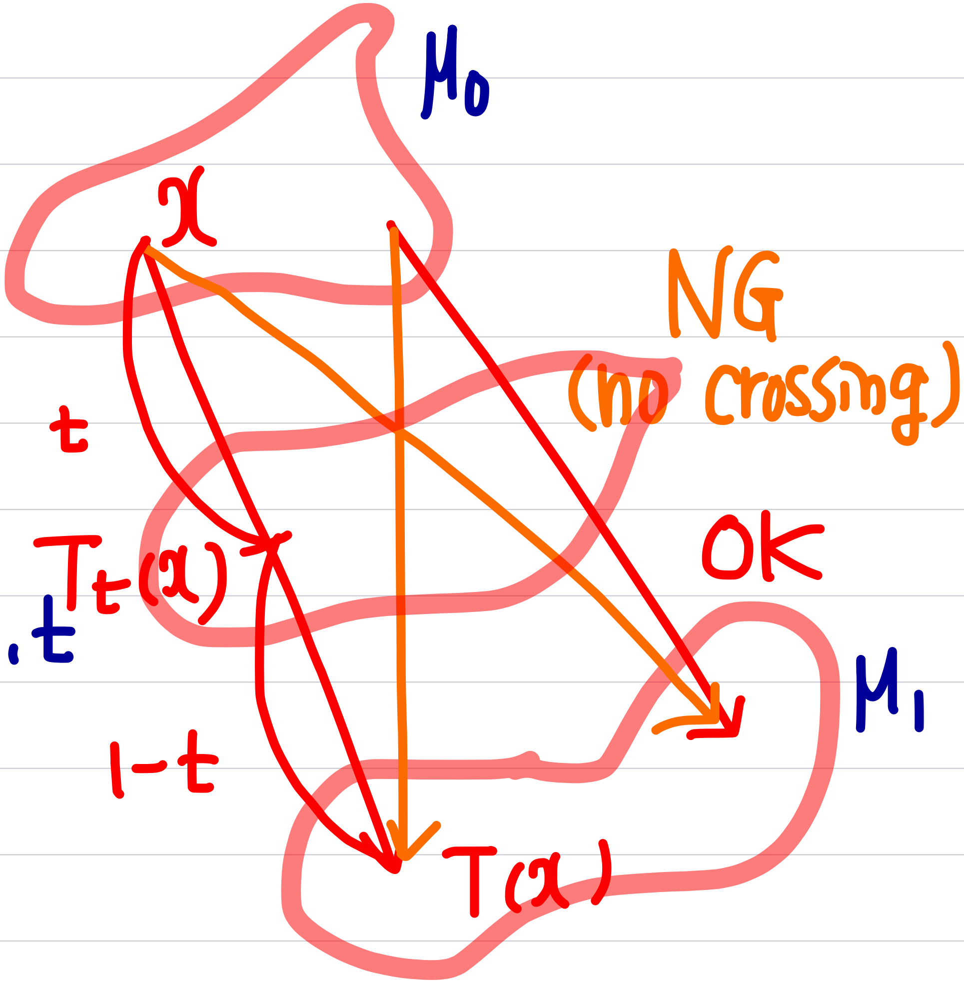
1. $\exists ! \, T \in \mathcal{T}(\mu_0, \mu_1) : \text{min.}$

2. $\exists ! \, \phi : \mathbb{R}^n \to \mathbb{R} : \text{convex}$ s.t.

$$T = \nabla \phi \quad \mu_0\text{-a.e.}$$

3. $T_t(x) := (1-t)x + tT(x)$

$$\{\mu_t := T_{t\#}\mu_0\}_{t \in [0,1]} : W_2\text{-geod from } \mu_0 \text{ to } \mu_1.$$

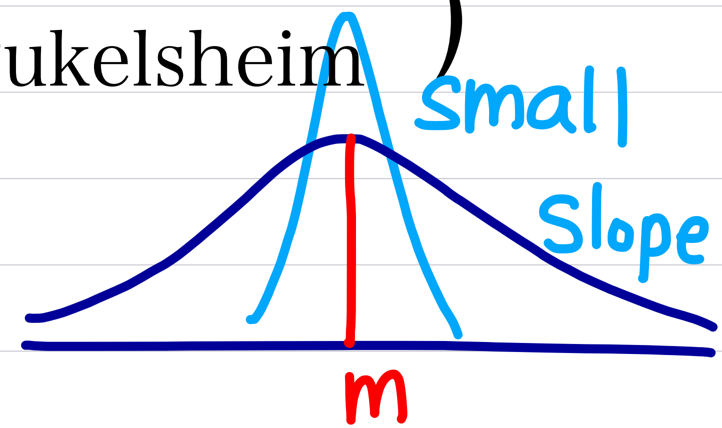


Rem. $\delta_{(1-t)x_0 + tx_1} : W_2\text{-geod}$ $\left(\text{NOT } (1-t)\delta_{x_0} + t\delta_{x_1}\right)$

Ex. (80': Dowson&Landau, Givens&Shortt, Knott&Smith, Olkin&Pukelsheim)

$N_i := N(m_i, V_i)$: Gauss

mean $\mathbb{R}^n$ $\quad$ S^n_+ cov.



$$X := V_1^{\frac{1}{2}}\left(V_1^{\frac{1}{2}} V_0 V_1^{\frac{1}{2}}\right)^{-\frac{1}{2}} V_1^{\frac{1}{2}} \in S^n_+$$

$$T(x) := X(x - m_0) + m_1 \in \mathcal{T}(N_0, N_1) : \text{min.}$$

$$W_2(N_0, N_1)^2 = \int_{\mathbb{R}^n} |x - T(x)|^2 \, dN_0(x)$$

$$= |m_0 - m_1|^2$$

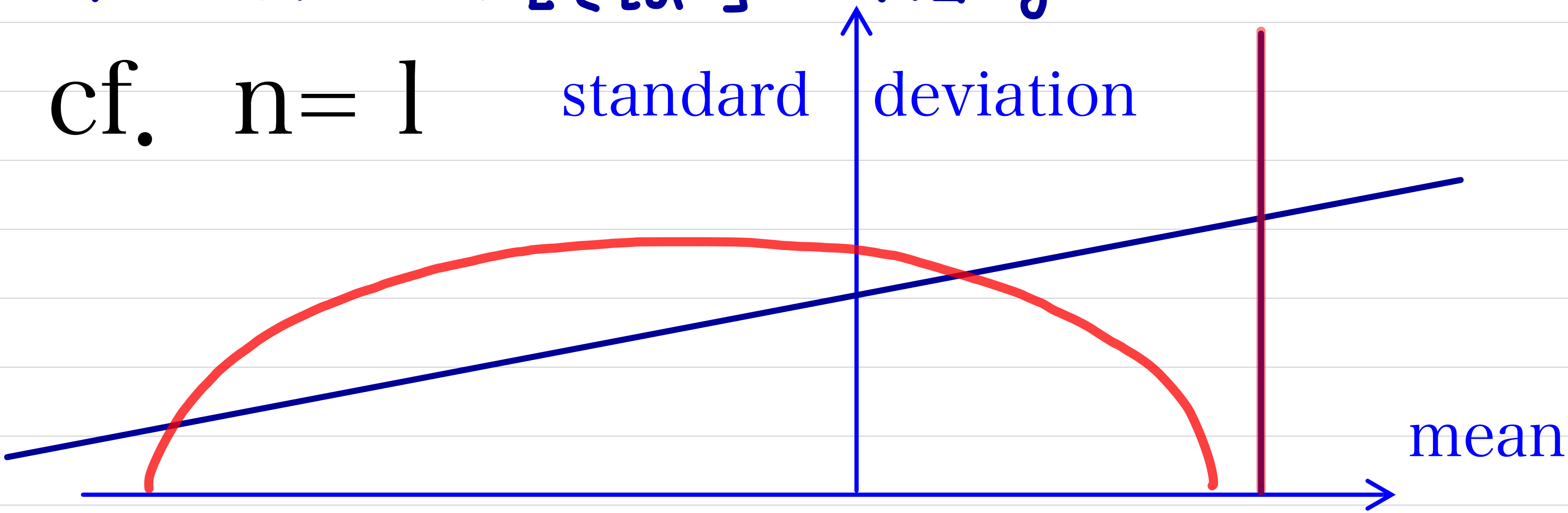
$$+ \operatorname{tr}\left(V_0 + V_1 - 2\sqrt{V_1^{\frac{1}{2}} V_0 V_1^{\frac{1}{2}}}\right).$$

$$m_t := (1-t)m_0 + t m_1$$

$$V_t := (I_n + t(X - I_n)) V_0 (I_n + t(X - I_n))$$

$\{N(m_t, V_t)\}_{t\in[0,1]}$: W_2-geod.

cf. $n=1$



Wasserstein : non-neg. curved.

Fisher : non-pos. curved.

2. ϕ-Gauss

$\varphi \in C(0,\infty)_+$: non-dec.

1. φ - exp. distrb w/ $m \in \mathbb{R}^n$ & $V \in S^n_+$.

   sol. to $\dot{y} = \varphi(y)$

2. min. of $E_\varphi(f) := \int_{\mathbb{R}^n} U_\varphi(f(x))\,dx$

$$U_\varphi(r) := \int_0^r \ln_\varphi(t)\,dt \quad : \text{cvx}$$

$$\ln_\varphi(t) := \exp_\varphi^{-1}(t)$$

$$= \int_1^t \frac{1}{\varphi(s)}\,ds$$

Ex. $\varphi(s) = s^q$

$$\exp_\varphi(\tau) := \max\{0,\ 1+(1-q)\tau\}^{\frac{1}{1-q}}$$

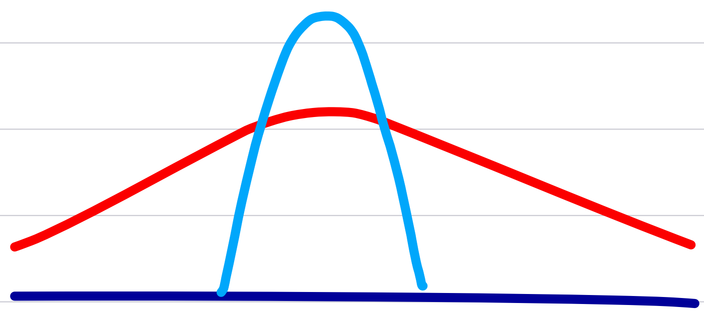
$^\forall q \in \left(0, \frac{n+4}{n+2}\right) =: Q_n \qquad (0^a := \infty \quad {}^\forall a<0)$

$^\exists c, \lambda \in C(S^n_+)$ s.t.

$$h_\varphi^{m,V}(x) := \exp_\varphi\Big(c(V) - \lambda(V)\underbrace{|x-m|_V^2}_{= \langle x-m,\ V^{-1}(x-m)\rangle}\Big)$$

1. $h_\varphi^{m,V}$: min. of E_φ.
2. $\{N_\varphi^{m,V} := h_\varphi^{m,V}\mathcal{L}^n\} \subset \mathcal{P}_2$.

$q \gtrless 1$: fat tail / spt cpt



Thm. (Takatsu, 2013)

If $\exists \delta \in Q_n$ s.t. $\dfrac{s^{\delta+\varepsilon}}{\varphi(s)} \xrightarrow[s\uparrow\infty]{s\downarrow 0} \begin{matrix} 0 \\ \infty \end{matrix}$ $\forall \varepsilon > 0$

then $\exists\, c, \lambda \in C(S_+^n)$ s.t.

1. $n_\varphi^{m,v}$: min. of E_φ.

2. $\{N_\varphi^{m,v}\} \subset (\mathcal{P}_2, W_2)$: cvx

$\Leftrightarrow \varphi(s) = s^q$

cf.

$\{G_\varphi^{m,v} = g_\varphi^{m,v}\mathcal{L}^n\} \subset (\mathcal{P}_2, W_2)$: cvx.

$g_\varphi^{m,v}(x) := \exp_\varphi\left(c(I_n) - \lambda(I_n)|x-m|_v^2\right)(\det v)^{-\frac{1}{2}}$.

$$E_\varphi^{m,V}(f) := E_\varphi(f) + \lambda(V)\int_{\mathbb{R}^n} |x-m|_V^2\, f(x)\,dx.$$

$$D_\varphi^{m,V}(f,g) := E_\varphi^{m,V}(f) - E_\varphi^{m,V}(g)$$

if $g = n_\varphi^{m,V}$

$$= \int_{\mathbb{R}^n} \underbrace{\left(U_\varphi(f) - U_\varphi(g) - U_\varphi'(g)(f-g)\right)}_{\geq 0} dx$$

U_φ : cvx

"Thm." (Ohta-T, 2013)

If $\sup_{s>0} \frac{s\varphi'(s)}{\varphi(s)} \in [0, 1+\frac{1}{n}] \setminus \{\frac{3}{2}\}$,

then $W_2(f\mathcal{L}^n, N_\varphi^{m,V}) \leq \sqrt{\frac{K(V)}{\lambda(V)} D_\varphi^{m,V}(f, n_\varphi^{m,V})}$

$K(V) := \max\{\text{e.v. of } V\}$

$\forall f\mathcal{L}^n \in \mathcal{P}_2$ w/ $\mathrm{spt}(f) \subset \mathrm{spt}(n_\varphi^{m,V})$.

transport ineq. → deviation ineq.

estimate $\alpha_\mu(r) := \sup_F \{ \mu(F \geq m_F + r) \}$

1-Lip. median.

Ex. $\mu = N_\varphi^{m,v}$, $\varphi(s) = s^q$

$A(p) := \int_{\mathbb{R}^n} (n_\varphi^{m,v})^p dx < \infty \quad \forall p > 1/2$

$$\Rightarrow \alpha_\mu^{q-p}(r) \ln_\varphi(2\alpha) \leq -A\left(\frac{p-q}{p-1}\right)^{1-p} \left[\left(\sqrt{\frac{(2-q)\lambda(v)}{K(v)}} r - \sqrt{A(2-q)} \right)^2 - A(2-q) \right]$$

for $q \in (0, 1+\frac{1}{n}] \cap (0, \frac{3}{2})$ & $q \neq 1$.

$p \in (\max\{2q-1, 1\}, 2]$.